EXERCISE 1 (10 + 10 + 5)

1. M is $f^{-1}(0)$ for $f(x,y) = y^2 - x^3 + x = y^2 - x(x^2-1)$ which is $f:\mathbb{R}^2 \to \mathbb{R}$ smooth (since it is a polynomial)

We just need to check that 0 is not a critical value for f:

$$df|_x = 2y\,dy - 3x^2 dx + dx = 2y\,dy - (3x^2-1)\,dx$$

$df|_x = 0$ iff $2y = 0$ so $y = 0$

$3x^2 - 1 = 0$ so $x = \pm\sqrt{\frac{1}{3}}$

however $f\left(\pm\sqrt{\frac{1}{3}}, 0\right) = \mp\sqrt{\frac{1}{3}}\left(\frac{1}{3} - 1\right) \neq 0$ so no critical point is in the preimage of 0.

Thus M is a smooth manifold of dim $2-1 = 1$ by the regular levelset thm.

2. $d\pi|_{(x,y)} = dx$ so critical values include values whose preimages contain points that have tangent lines parallel to $\frac{\partial}{\partial y}$.

But we need to study it on TM which we can compute by Prop 3.2.16 as $\operatorname{Ker} df_{(x,y)}$

$$T_{(x,y)}M = \left\{ v_1 \frac{\partial}{\partial x} + v_2 \frac{\partial}{\partial y} \;\middle|\; 2y\,v_2 - (3x^2-1)v_1 = 0 \right\}$$
$$= \left\{ v = (v_1, v_2) \;\middle|\; \langle v, (3x^2-1, 2y) \rangle = 0 \right\}$$

So $d\pi\left(T_{(x,y)}M\right) = \left\{ v_1 \;\middle|\; v_1 \frac{\partial}{\partial x} + v_2 \frac{\partial}{\partial y} \in T_{(x,y)}M \right\}$ so we have a critical point of $d\pi$ whenever $v_1 = 0$

for all vectors on $T_{(x,y)}M$.

$v_1 = 0 \Rightarrow y = 0$ then $(x,0)$ are critical pts of π

but $(x,0) \in M$ if $f(x,0) = -x(x^2-1) = 0$

⊛ and thus those pts are $(1,0), (0,0), (-1,0)$

and the critical values are $1, 0, -1$

Alternative only using multivariable analysis

- Argue that T_pM should be parallel to $\partial/\partial y$, study the fn to see where it is the case & justify w. the picture
- crit points $p \in M$ are given by $d(\pi|_M)|_{(x,y)} = 0$ and

since $(d\pi|_M) = (d\pi|_{(x,y)})|_{T_{(x,y)}M}$ and

for same reasons as above $T_{(x,y)}M = \operatorname{Ker}(df|_{(x,y)})$ so

$$\operatorname{Ker}(d\pi|_{(x,y)}) \supset \operatorname{Ker}(df|_{(x,y)})$$

and this, reasoning about euclidean vectors, means

$$\operatorname{rank}\begin{pmatrix} Df \\ \nabla\pi \end{pmatrix}\Big|_{(x,y)} = 1 \quad \text{implying } y = 0.$$

those kernels are $\perp$ to $\nabla\pi$ and ∇f and dim of LHS is $\leq$ dim of RHS $\Rightarrow$ need to look at when grads are parallel.

The rest follows as ⊛

3. If $P^{-1}(0)$ is a critical value of P this cannot be the case

$$dP = \frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy, \quad P(x,y) = xy \text{ in such that } P(0,0) = 0$$

and $dP|_{(0,0)} = 0$ gives a counterex.

(There are very many of them)

EXERCISE 2 (10 + 5 + 10)

1. An (r,s)-tensor is a multilinear map

$$\tau : \underbrace{V^* \times \cdots \times V^*}_{r \text{ copies}} \times \underbrace{V \times \cdots \times V}_{s \text{ copies}} \longrightarrow \mathbb{R}$$

They generalize the concept of vectors $V^* \to \mathbb{R}$ (since $V \simeq V^{**}$) and covectors $V \to \mathbb{R}$. One can think of them as multidimensional matrices.

The space of (r,s)-tensors and of (s,r)-tensors is n^{r+s} dimensional. and given a basis $\{e_i\}$ of V and $\{\varepsilon^i\}$ of V^* they have the form

$$\tau = \tau^{j_1 \dots j_r}_{i_1 \dots i_s} \, e_{j_1} \otimes \cdots \otimes e_{j_r} \otimes \varepsilon^{i_1} \otimes \cdots \otimes \varepsilon^{i_s}$$

and they are the building blocks to define differential forms (after antisymmetrization) and also inner products (symmetric, positive def., (0,2)-tensors)

(r,s)-Tensor fields are sections of the (r,s)-tensor bundle on a manifold M that is, smooth functions that send a point $p \in M \longmapsto \tau_p \in T^r_s(T_pM) =: T^r_s M$ so in this sense they are functions that pointwise produce tensors of the type above on the vector space T_pM

Then:

- antisymmetric $(0,s)$-tensor fields are the diff. forms
- symm, pos. definite, $(0,2)$-tensor fields are the metrics

2. (a) $$g(X,Y) = \sum_{j=1}^{2} dx^j\left(X^i \frac{\partial}{\partial x^i}\right) \otimes dx^j\left(Y^i \frac{\partial}{\partial x^i}\right)$$

$$= X^1 Y^1 + X^2 Y^2 \text{ "}= \langle x, y \rangle_{\mathbb{R}^2}\text{"}$$

(b) $$\phi^* g = \sum_{j=1}^{2} d\phi^* x_j \otimes d\phi^* x_j$$

$$= d(\rho\cos\theta) \otimes d(\rho\cos\theta) + d(\rho\sin\theta) \otimes d(\rho\sin\theta)$$

$$= (\cos\theta\, d\rho - \rho\sin\theta\, d\theta) \otimes (\text{same} \text{ —})$$
$$+ (\sin\theta\, d\rho + \rho\cos\theta\, d\theta) \otimes (\text{same} \text{ —})$$

$$= \cos^2\theta\, d\rho \otimes d\rho - \rho\cos\theta\sin\theta\,(d\rho \otimes d\theta + d\theta \otimes d\rho)$$
$$+ \rho^2 \sin^2\theta\, d\theta \otimes d\theta + \sin^2\theta\, d\rho \otimes d\rho$$
$$+ \rho\cos\theta\sin\theta\,(d\rho \otimes d\theta + d\theta \otimes d\rho)$$
$$+ \rho^2\cos^2\theta\, d\theta \otimes d\theta$$

$$= d\rho \otimes d\rho + \rho^2\, d\theta \otimes d\theta \quad \text{since } \sin^2 + \cos^2 = 1$$

EXERCISE 3 (8 pts)

for all $\varphi \in C^\infty(M)$

$$[fX, gY]\varphi = fX(gY\varphi) - gY(fX\varphi)$$

$$= f(Xg)Y\varphi + fg\,X(Y\varphi) - g(Yf)X\varphi - gfYX\varphi$$

$$= fg[X,Y]\varphi + \left(f(Xg)Y - g(Yf)X\right)\varphi$$

from which the claim follows

EXERCISE 4 (8+10+14)

1. $$\mathcal{L}_{X_H}\eta = d(\iota_{X_H}\eta) + \iota_{X_H}(d\eta)$$

$$= d(-H) + \iota_{X_H}(d\eta)$$

can also be done in coordinates but it is harder

$$= -dH + dH - H\left(\frac{\partial}{\partial z}\right)\eta = -H\left(\frac{\partial}{\partial z}\right)\eta$$

2. $$d\eta = -dy_i \wedge dx^i = dx^i \wedge dy_i$$

$$\Rightarrow (d\eta)^n = \underbrace{(dx^{i_1}\wedge dy_{i_1})\wedge \dots \wedge (dx^{i_n}\wedge dy_{i_n})}_{(*)}$$

$= n!\ dx^1 \wedge dy_1 \wedge \dots \wedge dx^n \wedge dy_n$

since $i_k = i_l \Rightarrow (*) = 0$ due to antisymmetry

every flipping due to the even number of variables requires an even number of flips & thus $+1$ terms and there are

$n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot 1$ ways of ordering the pairs $dx^i \wedge dy_i$

$$\Rightarrow (d\eta)^n \wedge \eta = n!\ dx^1 \wedge dy_1 \wedge \dots \wedge dx^n \wedge dy_n \wedge \overbrace{(dz - y_i dx^i)}$$

$$= n!\ dx^1 \wedge dy_1 \wedge \dots \wedge dx^n \wedge dy_n \wedge dz$$

since already all dx^i and dy_i occur in $(d\eta)^n \Rightarrow \in \Omega^{2n+1}$ w cst $n! \neq 0$ coeff $\Rightarrow$ volume form.

3. Since $\gamma'(t) = \dot{x}^i(t) \frac{\partial}{\partial x^i} + \dot{y}^i(t) \frac{\partial}{\partial y^i} + \dot{z}(t) \frac{\partial}{\partial z}$ we only need to compute X_H and compare term by term to check this.

$X_H = a^i \frac{\partial}{\partial x^i} + b_i \frac{\partial}{\partial y_i} + c \frac{\partial}{\partial z}$ for some C^∞ fns a^i, b_i, c

(1) $\iota_{X_H}(\eta) = c - y_i a^i = -H$

(2) $\iota_{X_H}(d\eta) = \iota_{X_H}(-dy_i \wedge dx^i) = \iota_{X_H}(dx^i \otimes dy_i - dy_i \otimes dx^i)$

$= -b_i dx^i + a^i dy_i$

$$= dH - dH\left(\frac{\partial}{\partial z}\right)\eta$$

$$\left(\begin{array}{l} dH = \dfrac{\partial H}{\partial x^i} dx^i + \dfrac{\partial H}{\partial y_i} dy_i + \dfrac{\partial H}{\partial z} dz \\[2ex] dH\left(\dfrac{\partial}{\partial z}\right) = \dfrac{\partial H}{\partial z} \end{array} \right)$$

$$= \frac{\partial H}{\partial x^i} dx^i + \frac{\partial H}{\partial y_i} dy_i + \underline{\frac{\partial H}{\partial z} dz}$$

$$\underline{-\frac{\partial H}{\partial z} dz} + \frac{\partial H}{\partial z} y_i dx^i$$

$$= \left(\frac{\partial H}{\partial x^i} + \frac{\partial H}{\partial z} y_i\right) dx^i + \frac{\partial H}{\partial y_i} dy_i$$

So, from (2),

$$a_i = \frac{\partial H}{\partial y_i} \quad \text{and} \quad b_i = -\left(\frac{\partial H}{\partial x^i} + \frac{\partial H}{\partial z}\right)$$

Thus (1) gives

$$c - y_i \frac{\partial H}{\partial y_i} = -H$$

That is $c = y_i \dfrac{\partial H}{\partial y_i} - H$

Therefore

$$(X_H)_{\gamma(t)} = \frac{\partial H}{\partial y_i}\Big|_{\gamma(t)} \frac{\partial}{\partial x^i} - \left(\frac{\partial H}{\partial x^i}\Big|_{\gamma(t)} + \frac{\partial H}{\partial z}\Big|_{\gamma(t)} \right) \frac{\partial}{\partial y_i}$$
$$+ \left(y_i(t) \frac{\partial H}{\partial y_i}\Big|_{\gamma(t)} - H\big|_{\gamma(t)} \right) \frac{\partial}{\partial z}$$

Comparing term by term gives the answer.